JOURNAL OF ALGEBRA 19, 31–50 (1971)

On Classifying Simple Lie Algebras of Prime Characteristic by Nilpotent Elements

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Communicated by Nathan Jacobson

Received February 3, 1970

1. INTRODUCTION

This paper presents results on the classification of simple Lie algebras of prime characteristic. While the simple Lie algebras over the complex numbers have been determined, as have their analogs over a field of prime characteristic (Seligman [14]), there are large classes of simple algebras which are not included in the latter class. These algebras, said to be of nonclassical type, do not possess a nondegenerate trace form. Consequently, the techniques employed by Seligman are not applicable. Important work in the classification of these algebras has been done by A. I. Kostrikin [10, and earlier papers] and by Kostrikin and Shafarevich [12, 13].

In considering the known nonclassical simple Lie algebras, Kostrikin has noted that all contain an element of index of nilpotence two. If x is an element of a Lie algebra L and $L(\mathrm{ad}\ x)^k = 0$ but $L(\mathrm{ad}\ x)^{k-1} \neq 0$, then x is said to be nilpotent of index k. A significant property of algebras containing an element of index of nilpotence two (such algebras are called strongly degenerate) is that they have no nondegenerate trace form. As has developed, it is an equivalent condition which has provided a tool for classifying algebras of nonclassical type. By a filtration of a Lie algebra L is meant a sequence of subalgebras

$$L = L_{-1} \supset L_0 \supset L_1 \supset \cdots \supset L_r \supset L_{r+1} = 0,$$

where L_0 is a given proper subalgebra and the remaining algebras are inductively defined by $L_{i+1} = \{x \in L_i \mid [x, L] \subset L_i\}$. A filtration is said to long if $r \geqslant 2$, which is true if and only if L is strongly degenerate.

Although there are no finite-dimensional complex analogs of the algebras

* The results presented here are contained in the author's doctoral dissertation submitted to the University of Illinois, Urbana, in 1969.

of nonclassical type, Kostrikin and Shafarevich have noted that the infinite Lie algebras of Cartan type over the complex numbers [3, 4, 6] correspond in a natural fashion to the known restricted simple Lie algebras of nonclassical type. The correspondence involves replacing the complex numbers by an algebraically closed field of prime characteristic and factoring out an appropriate ideal. Just as a more general idea of filtration yields much information on the algebras of Cartan type, Kostrikin and Shafarevich have used techniques based upon the filtration to prove limited versions [12, Theorem 1; 13, Paragraph II, Section 1] of their conjecture that the known simple restricted algebras, in fact, exhaust all such algebras. Utilizing a similar technique, Wilson [15] has generalized this to obtain a unified means of describing all the known nonclassical simple Lie algebras.

Our first result is a contribution toward establishing the existence of a long filtration in the simple Lie algebras of nonclassical type. It is a generalization of Kostrikin's principle result in [10], which is formulated as follows: If L is a simple Lie algebra over an algebraically closed field of characteristic $p > 5$ which satisfies

(i) L is restricted, i.e., $\operatorname{ad} L$ is closed under p-th powers (restricted Lie algebras are also called Lie p-algebras),

(ii) L is not strongly degenerate,

(iii) L has a Cartan subalgebra H, $L = H + \sum_{\alpha \neq 0} L_\alpha$, for which $L(\operatorname{ad} a)^{p-1} = 0$ for some nonzero a in H or in some root space L_α,

then L is of classical type.

If, as in (iii), an element a is either in L_α for some α or in H, it is called H-uniform. In this paper, we show that deleting (i) does not alter the conclusion. Our first main result is the following:

THEOREM A. *Let L be a simple algebra over an algebraically closed field F of characteristic $p > 5$. Let H be a Cartan subalgebra of L for which* (ii) *and* (iii) *are satisfied. Then L is of classical type.*[1]

The remainder of the paper is devoted to two results on algebras of low dimension relative to p. Suppose L is a simple Lie algebra over an algebraically closed field of characteristic $p > 5$. Kostrikin has shown that L is of classical type if $\dim L < p$, and he has some partial results [8] for $\dim p$. In Section 6 we use Theorem A and several results of Kostrikin to show that if L has $\dim p$, then it is either the nonclassical p-dimensional Witt algebra or an algebra of classical type (Theorem B). The final theorem, which proves a conjecture of Kostrikin [11], is the following:

[1] In *Bull. Amer. Math. Soc.* **76** (1970), 393, the author stated this result incorrectly. The correct version is given here.

THEOREM C. *Suppose L is a semisimple Lie algebra over an algebraically closed field F of prime characteristic p which admits a faithful representation Γ of degree $n < p - 1$. Then L is a direct sum*

$$L = L_1 \oplus \cdots \oplus L_r ,$$

where each L_i is a simple algebra of classical type.

Kostrikin has proved this under the additional hypotheses that L is a p-algebra and Γ is a p-representation.

Zassenhaus has indicated that Theorem C might have applications in the theory of simple groups.

In [13], Theorems A and C are reported to have been proved by Zerebeovym.

2. PRELIMINARY MATERIAL

Unless otherwise stated, all Lie algebras considered are assumed to be finite-dimensional over a field of prime characteristic. As in the introductory remarks, F will denote an algebraically closed field of prime characteristic p, generally with $p > 5$. The prime subfield of F will be denoted by P. By R_a is meant the adjoint endomorphism of a, i.e., $bR_a = [b, a]$. We write

$$[xy_1^{k_1} \cdots y_s^{k_s}] = xR_{y_1}^{k_1} \cdots R_{y_s}^{k_s} .$$

It follows from easy induction arguments that

$$[R_x R_y{}^n] = R_{[xy^n]} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} R_y{}^i R_x R_y^{n-i}.$$

The above is frequently used to express $[x[yz^n]]$ as the sum

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} [xz^i y z^{n-i}].$$

Henceforth, let L be a Lie algebra over F. If L is centerless, $R_a = 0$ if and only if $a = 0$, and the following identities are valid (see Section 2 of [7]).

(1) $R_a{}^k = 0 \Rightarrow R_{[xa^{k-1}]}^{k-1} = 0, \qquad 4 \leqslant k \leqslant p - 1.$

(2) $R_b{}^3 = 0 \Rightarrow R_b{}^2 R_x R_b = R_b R_x R_b{}^2, \qquad$ or $\qquad R_b R_{[xb]} R_b = 0.$

(3) $R_b{}^3 = 0 \Rightarrow R_{[xb^2]}^3 = 0.$

(4) $R_b^3 = 0 \Rightarrow [R_b R_{[xb]}^m]^2 = R_b^2(R_x^2 R_b^2)^m$.

We recall the following definitions introduced by Kostrikin [8, 10].

DEFINITION. An element a in L is nilpotent with index of nilpotence k if $LR_a^k = 0$ but $LR_a^{k-1} \neq 0$.

DEFINITION. If L contains a nilpotent element with index of nilpotence two, L is strongly degenerate.

DEFINITION. If H is a Cartan subalgebra of L and $L = H + \sum_{\alpha \neq 0} L_\alpha$ is the corresponding Cartan decomposition, then $x \in L$ (a vector space $M \subset L$) is called H-uniform if either $x \in L_\alpha$ for some α, or $x \in H$ (respectively, $\sum_{\alpha \neq 0}(L_\alpha \cap M) + (H \cap M) = M$).

DEFINITION. $L(m) = \{c \in L \mid R_c R_x^k R_c = 0 \text{ for } x \text{ in } L,\ 1 \leqslant k \leqslant 2m - 1\}$. This definition will be used in the proof of Theorem B, for which the following theorem, found in [8], is also needed:

THEOREM (Kostrikin). *If L is strongly degenerate, $L((p-3)/2) \neq \{0\}$.*

3. PROOF OF THEOREM A

Fix a Cartan subalgebra H of L satisfying the hypothesis of Theorem A. When there is no confusion concerning the Cartan subalgebra H in question, H-uniform elements (subspaces) will be called uniform. The proof of the theorem follows the general outline of the Kostrikin result in [10] which was mentioned in the Introduction. The argument in [10] divides into four parts. The first and third parts do not specifically require that L be restricted. The deletion of this hypothesis in part two necessitates the use of different techniques to reach the same conclusion. Part four of the proof requires a slight modification to eliminate the possibility of L being an Albert–Zassenhaus algebra.

In order to establish necessary tools, we will state the results of the first part of Kostrikin's proof. The basic result of this part is formulated below. It will be assumed that all the hypotheses of Theorem A hold, although not all are necessary for this result.

PROPOSITION 3.1 (Kostrikin). *In L there exists a uniform element $b \neq 0$ such that*

1) $R_b^3 = 0$;

2) *if x is a nonzero uniform element with* $R_x^3 = 0$, *then* $\dim LR_x^2 \geqslant \dim LR_b^2$;

3) $\dim LR_b^2 = p^l$ *for some* $l \geqslant 0$ (*we write* $r = p^l$);

4) *there is a nonzero uniform element a with* $R_a^3 = 0$ *which generates, together with b, a three-dimensional subalgebra* $M = \langle a, b, h\rangle$, $h = [ab]$, $[ah] = 2a$, $[bh] = -2b$;

5) *as a vector space, L allows a decomposition into a direct sum of uniform subspaces,*

$$L = \overline{M} + M_1 + M_{-1} + M_0 ,$$

where $\overline{M}$ is spanned by linearly independent uniform elements

$$a_i , a_iR_b , a_iR_b^2, \qquad i = 1, 2,..., r, \qquad (a_1 = a).$$

All a_i lie in different root spaces L_{α_i},

$$a_iR_h = 2a_i , \qquad R^3_{a_i} = 0;$$

$\dim M_1 = \dim M_{-1}$; $xR_h = x$, $yR_h = -y$, $zR_h = 0$ *for all x in* M_1, *y in* M_{-1}, *z in* M_0 ;

$$M_{-1}R_a = M_1 , \ M_1R_b = M_{-1} , \ (M_1 + M_0) R_a = 0, \ (M_{-1} + M_0) R_b = 0.$$

For the proof of this proposition, the reader is referred to [10] as restrictedness is not necessary.

However, it is instructional to consider the decomposition alluded to in 5). Using the b described in 1) and 2), we construct a vector space decomposition of L in the following manner. Let M_2 be a vector space complement of $\ker R_b^2$ in L, and define $\overline{M}_0 = M_2R_b$ and $M_{-2} = M_2R_b^2 = LR_b^2$. Let M_1 be a vector space complement of $\overline{M}_0 + \ker R_b$ in $\ker R_b^2$, and define $M_{-1} = M_1R_b$. Finally, let M_0 be a vector space complement of $M_{-1} + M_{-2}$ in $\ker R_b$. Obviously, these subspaces of L are not unique. However, since b is uniform, M_2 , M_1 , and M_0 may be chosen so that they are uniform. Assuming this to be the case, all the subspaces will be uniform. Thus, L is decomposed into a direct sum of uniform subspaces

$$\begin{aligned} L &= M_2 + \ker R_b^2 = M_2 + M_1 + \overline{M}_0 + \ker R_b \\ &= M_2 + M_1 + \overline{M}_0 + M_{-1} + M_{-2} + M_0 . \end{aligned} \tag{5}$$

Associated with the subspaces in this decomposition are the subsets $\Omega_{\pm 2}$, $\Omega_{\pm 1}$, Ω_0 , $\overline{\Omega}_0$ of the set Ω of the roots of L with respect to H. These subsets are defined by the relations $M_i = \sum_{\omega\in\Omega_i} (L_\omega \cap M_i)$ for $i = 0, \pm 1,$

± 2, and $\overline{M}_0 = \sum_{\omega \in \overline{\Omega}_0} (L_\omega \cap \overline{M}_0)$. All the intersections under the summation signs are assumed to be nontrivial.

In the notation of part 5) of the proposition, $\overline{M} = M_2 + \overline{M}_0 + M_{-2}$, where $a_i \in M_2$, $a_i R_b \in \overline{M}_0$, and $a_i R_b{}^2 \in M_{-2}$. Part 5) further indicates that the subscripts of the subsets Ω_i, $i = 0, \pm 1, \pm 2$, and $\overline{\Omega}_0$ arise from the action of the roots on h, which acts diagonally. In particular, for $\omega \in \Omega_i$ we have $\omega(h) = i$. Consequently, these subsets are disjoint with the possible exception of Ω_0 and $\overline{\Omega}_0$. Since the a_i's are elements of different root spaces and they span M_2, we conclude that L_{α_i} is one-dimensional and the a_i's (except $a_1 = a$) are arbitrary nonzero elements of L_{α_i}. Finally, we state two lemmas which are needed later.

Lemma 3.2 (Kostrikin). *Let* $\alpha_{k_1}, \dots, \alpha_{k_m}$ *be any roots in* Ω_2. *Then* $j_1\alpha_{k_1} + \cdots + j_m\alpha_{k_m} \in \Omega_2$ *if and only if* $j_1 + \cdots + j_m = 1$.

Lemma 3.3 (Kostrikin). $LR^2_{[a_i b^2]} = LR_b{}^2$.

4. Determination of $M_2 + \overline{M}_0 + M_{-2}$

We now examine the subspaces determined by the nilpotent element b. For this it will be necessary to consider decompositions resulting from other nilpotent elements. Let d be a uniform element in L satisfying $R_d{}^3 = 0$. Let $M_i(d)$, $i = 0, \pm 1, \pm 2$, and $\overline{M}_0(d)$ denote a decomposition of L into uniform subspaces, as defined in (5), corresponding to the element d. Furthermore, let $\Omega_i(d)$, $i = 0, \pm 1, \pm 2$, and $\overline{\Omega}_0(d)$ denote the resulting sets of roots. Whenever the parentheses are deleted, the presence of b is implied. For example, $M_{-2} = M_{-2}(b)$.

Proposition 4.1. *Under the conditions of Proposition* 3.1, *the subspace* $\overline{M}$ *of* L *coincides with* M, *or, equivalently,* $\dim LR_b{}^2 = r = 1$.

For the proof of this proposition, our attention will be essentially confined to $\overline{M}$. Recall that $a_1 = a$ is fixed by Proposition 3.1, while the remaining a_i's have been chosen as arbitrary nonzero elements of L_{α_i}, $i = 2, 3, \dots, r$. The roots β of L whose corresponding root spaces L_β intersect $\overline{M}$ nontrivially are

$$\Omega_2 = \{\alpha_1, \dots, \alpha_r\}, \ \overline{\Omega}_0 = \{0, \alpha_2 - \alpha, \dots, \alpha_r - \alpha\},$$
$$\text{and } \Omega_{-2} = \{-\alpha, \alpha_2 - 2\alpha, \dots, \alpha_r - 2\alpha\}.$$

The latter two equalities arise from the definitions $M_2 R_b = \overline{M}_0$ and $M_2 R_b{}^2 = M_{-2}$. We begin the proof by examining the multiplication of elements in $\overline{M}$.

LEMMA 4.2. *The restriction of $R^2_{[a_jb^2]}$ to M_2 is a bijective mapping onto M_{-2} for $j = 1, 2,..., r$. The same is true for $R^2_{a_j}$ with M_2 and M_{-2} interchanged.*

Proof. The first statement will be proved. The proof for $R^2_{a_j}$ follows by symmetry. From identity (3) we conclude $R^3_{[a_jb^2]} = 0$. Thus, the root spaces $M_{\pm 2}([a_jb^2])$ and the sets of roots $\Omega_{\pm 2}([a_jb^2])$ can be defined. It is sufficient to show that $M_2 = M_2([a_jb^2])$ and $M_{-2} = M_{-2}([a_jb^2])$ because by construction $R^2_{[a_jb^2]}$ is a bijective mapping from $M_2([a_jb^2])$ onto $M_{-2}([a_jb^2])$. From the equality $LR_b{}^2 = LR^2_{[a_jb^2]}$ of Lemma 3.3 it follows that $\Omega_{-2} = \Omega_{-2}([a_jb^2])$ and, in particular, that $LR^2_{[a_jb^2]} \cap L_\gamma = \{0\}$ unless $\gamma \in \Omega_{-2}$. Thus, $R^2_{[a_jb^2]}$ must annihilate all uniform elements excepting possibly those in L_β, where

$$\begin{aligned}\beta \in \{&-\alpha - 2(\alpha_j - 2\alpha), \alpha_2 - 2\alpha - 2(\alpha_j - 2\alpha),..., \alpha_r - 2\alpha - 2(\alpha_j - 2\alpha)\}\\ &= \{3\alpha - 2\alpha_j , \alpha_2 + 2\alpha - 2\alpha_j ,..., \alpha_r + 2\alpha - 2\alpha_j\} = \Omega_2([a_jb^2]).\end{aligned}$$

Since the sum of the coefficients of each element in the above set equals one, by Lemma 3.2 we conclude that $\Omega_2([a_jb^2]) \subset \Omega_2$. Using the fact that $\dim L_\gamma = 1$ for γ in $\Omega_2 \cup \Omega_{-2}$, we obtain that

$$|\,\Omega_2([a_jb^2])| = \dim LR^2_{[a_jb^2]} = \dim LR_b{}^2 = r,$$

so $\Omega_2([a_jb^2]) = \Omega_2$. Finally, the aforementioned dimension argument and the equalities $\Omega_{\pm 2} = \Omega_{\pm 2}([a_jb^2])$ imply that $M_{\pm 2} = M_{\pm 2}([a_jb^2])$, proving the lemma.

LEMMA 4.3. *The operator $A_{i,s} = R_{a_i}R^s_{[a_ib]}R_{a_i}$ is identically zero for odd s.*

Proof. The proof procedes by induction on s. For $s = 1$,

$$A_{i,1} = R_{a_i}R_{[a_ib]}R_{a_i} = 0 \qquad \text{(by (2) since } R^3_{a_i} = 0).$$

Suppose the equality is true for $s = 2k - 1$, $k > 1$, then

$$\begin{aligned}A_{i,2k+1} &= R_{a_i}R_{[a_ib]}(R_{a_i}R_b - R_bR_{a_i})\, R^{2k-1}_{[a_ib]}R_{a_i}\\ &= A_{i,1}R_bR^{2k-1}_{[a_ib]}R_{a_i} - R_{a_i}R_{[a_ib]}R_bA_{i,2k-1} = 0.\end{aligned}$$

LEMMA 4.4. *The derivation $R^p_{[a_ib]}$ is the sum of two derivations, R_{η_ih} and D_i, where $\eta_i \neq 0$ is an element of F and $\overline{M}D_i = 0$ for $i = 1, 2,..., r$.*

Proof. Initially, note that if x is an element of L_γ, then

$$xR^p_{[a_ib]} \in L_{\gamma+p(\alpha_i-\alpha)} = L_\gamma .$$

(That is, root spaces are invariant under $R^p_{[a_ib]}$.) Set $D_i = R^p_{[a_ib]} - R_{\eta_i h}$, where η_i in F is chosen so that $bD_i = 0$. Obviously, D_i is a derivation since $R^p_{[a_ib]}$ and $R_{\eta_i h}$ are.

We prove by induction that $[b[a_ib]^s] \neq 0$ for any $s \geq 1$. First observe that

$$[b[a_ib]] = -[a_ib^2] \neq 0.$$

Next, using the fact that $[ba_ib^2] \in \bar{M}_0 R_b{}^2 = 0$, we obtain

$$\begin{aligned}[b[a_ib]^2] &= [b(a_ib - ba_i)[a_ib]] = [ba_ib[a_ib]] \\ &= [ba_iba_ib] = [ba_i[ba_i]b] + [ba_i{}^2b^2] \\ &= [ba_i{}^2b^2] \neq 0,\end{aligned}$$

since, by Lemma 4.2, $R^2_{a_i}$ is an injective mapping from M_{-2} onto M_2. Suppose that $[b[a_ib]^l] \neq 0$ for some $l \geq 2$. Then

$$\begin{aligned}[b[a_ib]^{l+1}] &= [ba_i{}^2b^2[a_ib]^{l-1}] = [ba_i{}^2b^2a_iba_ib \cdots a_ib] \\ &= [ba_ib \cdots a_iba_i{}^2b^2] \qquad \text{(see (2))} \\ &= [b[a_ib]^{l-1}]\, R^2_{a_i} R_b{}^2 \neq 0.\end{aligned}$$

Thus, for $s = p$, $bR^p_{[a_ib]} \neq 0$. In particular, $\eta_i \neq 0$ since $bD_i = 0$.

To complete the proof, we show that $\bar{M}D_i = 0$. It is sufficient to show that $a_jD_i = 0$ for $j = 1, 2,..., r$, since

$$bD_i = 0 \qquad \text{and} \qquad \bar{M} = \langle a_j, a_jR_b, a_jR_b{}^2 \mid j = 1, 2,..., r\rangle.$$

Since L_{α_j} is one-dimensional and D_i maps each root space into itself ($R^p_{[a_ib]}$ and R_h have this property), it follows that $a_jD_i = \nu_{j,i}a_j$ for some $\nu_{j,i}$ in F. Restricting our attention to $[a_jb^2a_i]$, we have

$$\begin{aligned}[a_jb^2a_i]\, R_h &= [a_jb^2ha_i] + [a_jb^2[a_ih]] \\ &= -2[a_jb^2a_i] + 2[a_jb^2a_i] = 0,\end{aligned}$$

and, furthermore,

$$[a_jb^2a_i]\, D_i = (\nu_{j,i} + \nu_{i,i})[a_jb^2a_i].$$

Therefore,

$$\begin{aligned}0 = [a_jb^2]\, A_{i,p} &= [a_jb^2]\, R_{a_i}R^p_{[a_ib]}R_{a_i} = [a_jb^2a_i](D_i - \eta_iR_h)\, R_{a_i} \\ &= (\nu_{j,i} + \nu_{i,i})[a_jb^2a_i{}^2].\end{aligned}$$

Since $[a_jb^2a_i{}^2] \neq 0$, we must have $\nu_{j,i} + \nu_{i,i} = 0$. Setting $i = j$, we obtain $\nu_{i,i} = 0$, and thus $\nu_{j,i} = 0$, or $a_jD_i = 0$.

Included in the proof of Lemma 4.2 is the fact that $M_{\pm 2}([a_ib^2]) = M_{\pm 2}$ for $i = 2, 3,..., r$. While it can be easily shown that $\bar{\Omega}_0([a_ib^2]) = \bar{\Omega}_0$, the dimensionality of the nonzero root spaces which intersect $\bar{M}_0$ nontrivially is unknown. If it were one for each of the root spaces, then the equality $\bar{M}_0([a_ib^2]) = \bar{M}_0$ would follow, and a short argument would yield the fact that $\bar{M}$ is a subalgebra of L with multiplication $[\bar{M}_0, \bar{M}_0] = 0$. Although this is not necessarily the case, the following lemma provides sufficient information on $[\bar{M}_0, \bar{M}_0]$ to allow us to complete the proof of the proposition.

LEMMA 4.5. *For arbitrary $i, j = 1, 2,..., r$, $[a_jb[a_ib]\, a_i] = 0$.*

Proof. Since $R^p_{[a_ib]}$ acts as $\eta_i R_h$ on $\bar{M}$, it follows immediately that $[a_jb]\, R^p_{[a_ib]} = 0$. Hence, there is a minimal $k \leqslant p$ for which $[a_jb]\, R^k_{[a_ib]} = 0$. The identity $R_bR_{[a_ib]}R_b = 0$ (see (2)) allows us to write

$$[a_jb]\, R^k_{[a_ib]} = [a_jb[a_ib]\, a_ib \cdots [a_ib]\, a_ib] = 0,$$

if k is even and

$$[a_jb]\, R^k_{[a_ib]} = [a_jb[a_ib]\, a_ib \cdots [a_ib]] = 0$$

for k odd. We claim that in the first case the last b may be deleted. If $x = [a_jb]\, R^{k-1}_{[a_ib]} R_{a_i} \neq 0$, then $x \in M_2$ (add the coefficients of the α_i's and appeal to Lemma 3.2). Thus R_b will not annihilate x. Hence, in both instances it follows that $[a_jb]\, R^{l-1}_{[a_ib]} R_{a_i} \neq 0$, but $[a_jb]\, R^{l-1}_{[a_ib]} R_{a_i} R_b R_{[a_ib]} R_{a_i} = 0$, where $l = 2[(k-1)/2]$. Appealing once again to Lemma 3.2, we conclude that $[a_jb]\, R^{l-1}_{[a_ib]} R_{a_i}$ is a nonzero element of L_{α_s}, where $\alpha_s = \alpha_j + l(\alpha_i - \alpha)$. Since $a_jR^p_{[a_ib]} = 2\eta_i a_j \neq 0$, we have $0 \neq [a_j[a_ib]^l] \in L_{\alpha_s}$. This implies, by the one-dimensionality of L_{α_s}, that $[a_j[a_ib]^l\, b[a_ib]\, a_i] = 0$.

To complete the proof, it is sufficient to show that l may be increased to $l+1$, for doing so $p - l$ times yields

$$0 = [a_j[a_ib]^p\, b[a_ib]\, a_i] = 2\eta_i[a_jb[a_ib]\, a_i], \qquad \eta_i \neq 0.$$

This augmenting is accomplished by straightforward computation, using the identities $R_bR_{[a_ib]}R_b = 0 = R_{a_i}R_{[a_ib]}R_{a_i}$ ($R^3_{a_i} = 0$ by point 5) of Proposition 3.1) and the fact that $R_{[a_ib]}$ is a derivation.

$$\begin{aligned}
0 &= [a_j[a_ib]^l\, b[a_ib]\, a_i]\, R_{[a_ib]} \\
&= [a_j[a_ib]^{l+1}\, b[a_ib]\, a_i] + [a_j[a_ib]^l\, [b[a_ib]][a_ib]\, a_i] \\
&\quad + [a_j[a_ib]^l\, b[a_ib][a_i[a_ib]] = [a_j[a_ib]^{l+1}\, b[a_ib]\, a_i] \\
&\quad - [a_j[a_ib]^l\, [a_ib^2][a_ib]\, a_i] + [a_j[a_ib]^l\, b[a_ib][ba_i^2]] \\
&= [a_j[a_ib]^{l+1}\, b[a_ib]\, a_i] + 2[a_j[a_ib]^l\, ba_ib[a_ib]\, a_i] \\
&\quad - 2[a_j[a_ib]^l\, b[a_ib]\, a_iba_i] \\
&= [a_j[a_ib]^{l+1}\, b[a_ib]\, a_i] - 2[a_j[a_ib]^l\, [a_ib]\, b[a_ib]\, a_i] \\
&= -[a_j[a_ib]^{l+1}\, b[a_ib]\, a_i].
\end{aligned}$$

Let us assume that $\dim LR_b{}^2 > 1$ and choose a_2 such that $R^p_{[a_2b]} = R_h + D_2$. Returning to the Kostrikin proof, we will repeat the argument that this assumption leads to the contradiction that $R^2_{a_0} = 0$, where $a_0 = \sum_{i=0}^{p-1} 2^{-i}[a_2[a_2b]^i]$. (Lemma 2.4, [10].) Since $a_0 \in M_2$, it is sufficient to show that $M_{-2}R^2_{a_0} = 0$. Expanding $R^2_{a_0}$, we obtain a sum of terms of the form $[R_{a_2}R_b]^{i_1} A_{2,i_2}[R_{a_2}R_b]^{i_3}$. By Lemma 4.3, $A_{2,1} = 0$. For $[a_sb^2]$ in M_{-2}, $[a_sb^2]\, R^i_{[a_2b]} = \nu[a_kb^2]$, where $\alpha_k = \alpha_s + i(\alpha_2 - \alpha)$ (Lemma 3.2), whence the relations $R_bR_{[a_2b]}R_b = 0$ and $M_0R_{[a_2b]}R_{a_2} = 0$ imply

$$[a_sb^2]\, R^i_{[a_2b]}R_{a_2}R^2_{[a_2b]} = \nu[a_kb^2]\, R_{a_2}R^2_{[a_2b]}$$
$$= \nu\{[a_kba_2b[a_2b]^2] - [a_kb[a_2b]^3]\} = 0.$$

Finally, we note that when applied to M_{-2}, $R_{[a_2b]}R^2_{a_2} = R_{a_2}R_bR^2_{a_2} = R^2_{a_2}R_bR_{a_2} = -R^2_{a_2}R_{[a_2b]}$. Combining these results, we obtain

$$[a_sb^2]\, R^2_{a_0} = [a_sb^2] \sum \nu_{i_1,i_2,i_3} R^{i_1}_{[a_2b]}A_{2,i_2}R^{i_3}_{[a_2b]}$$
$$= [a_sb^2] \sum_{i,j=0}^{p-1} 2^{-i-j}(-1)^i\, R^i_{[a_2b]}R^2_{a_2}R^j_{[a_2b]}$$
$$= [a_sb^2] \sum_{i,j=0}^{p-1} 2^{-i-j}R^2_{a_2}R^{i+j}_{[a_2b]}\,.$$

Observing that for x in M_2 we have $x(2^{-p-k}R^{p+k}_{[a_2b]}) = 2^{-p-k}xR_hR^k_{[a_2b]} = 2^{-k}xR^k_{[a_2b]}$ and that for any integer k, $0 \leqslant k < p$, there are p pairs i, j for which $i + j \equiv k (\mathrm{mod}\, p)$, we conclude that $R^2_{a_0} = 0$. This concludes the proof of Proposition 4.1.

5. Completion of Proof

The remainder of the proof for Theorem A is contained in [10]. While the results there assume that L is restricted, careful examination of the proofs reveal that this is not necessary. We will outline the argument, indicating the source of the proofs.

Proposition 5.1 [10, Proposition 2.2.]. *Let L be a Lie algebra satisfying the hypothesis of Theorem A and let b be a nilpotent uniform element for which* $\dim LR_b{}^2 = 1$. *Then b gives rise to a decomposition of L into uniform subspaces as in* (5) *which satisfies the properties stated in Proposition* 3.1. *Furthermore,* *the subspace M_0 is spanned by elements of the form $b(R_{f_i}R_{f_j} + R_{f_j}R_{f_i})$ where f_i, f_j are uniform elements of M_1.*

In particular, this implies that every uniform element b in L satisfying $\dim LR_b^2 = 1$ is contained in a three-dimensional Lie subalgebra $M = \langle a, b, h\rangle$ of L with $a \in L_\alpha$, $h \in H$, $\alpha(h) \neq 0$. Moreover, $\dim L_{-\alpha} = \dim L_\alpha = 1$. If this were true for every uniform element b not in H, then the following theorem of Block [1] would complete the proof of Theorem A.

THEOREM (Block). *Let L be a simple finite-dimensional Lie algebra over F of characteristic $p > 5$. Suppose H is a Cartan subalgebra of L such that for every nonzero root α, L_α is one-dimensional and $\alpha([L_{-\alpha}, L_\alpha]) \neq 0$. Then L is either of classical type or of rank one, in which case L is an Albert–Zassenhaus algebra.*

The case where L is Albert–Zassenhaus is eliminated since Block's proof shows that H must be one-dimensional. We immediately conclude that $M_0 = 0$ since H is its own normalizer and that the roots of H are 0, 1, -1, 2, -2. The one-dimensionality of the root spaces implies $\dim L = 5$, contradicting the inequality $p > 5$.

In showing that the hypotheses of the above theorem are satisfied, we will first consider the two subspaces M_1 and M_{-1}. The basic results are summarized in the following proposition, which is essentially Proposition 3.1 in [10].

PROPOSITION 5.2. *For any root ω in $\Omega_1 \cup \Omega_{-1}$, $\dim L_\omega = 1$. Furthermore, $\Omega_1 = \{\alpha_1, ..., \alpha_s; \alpha - \gamma_1, ..., \alpha - \gamma_s\}$ and the basic elements f_ω for M_1 can be chosen so that $[f_{\alpha-\gamma_i}, f_{\gamma_i}] = a$, $1 < i < s$; $[f_{\omega_1}, f_{\omega_2}] = 0$ if $\omega_1 + \omega_2 \neq \alpha$; $\gamma_i([bf_{\alpha-\gamma_i} f_{\gamma_i}]) \neq 0$.*

In particular, we note that the hypotheses of Block's theorem are satisfied for the root spaces in M_1 and M_{-1}, implying that it remains only to consider the uniform elements of M_0 which do not belong to H. From each such root space L_δ, where $\delta \in \Omega_0$ and $\delta \neq 0$, we pick a nonzero element $x = [bf_i f_j]$. Proposition 5.2 guarantees that elements in L_δ are of this form where $f_k \in L_{\gamma_k}$, $k = i, j$; $[f_i f_j] = 0$, $-\alpha + \gamma_i + \gamma_j = \delta$. For this element x, we attempt to find some element z satisfying $\dim LR_z^2 = 1$ and $x \in M_1(z)$ or $x \in M_{-1}(z)$. The preceding proposition would then guarantee the existence of a uniform element y in $M_1(z)$ (or $M_{-1}(z)$) for which $\langle x, [xy], [zyx]\rangle$ is the three-dimensional simple algebra. The proof of the existence of such a z is given in [10] following the statement of Proposition 4.1. This concludes the proof of Theorem A.

6. Simple Lie Algebras of Prime Dimension

This section and the next contain applications of Theorem A. The following result completes the partial examination by Kostrikin [8] of the simple p-dimensional Lie algebras. There Kostrikin has shown that the only simple, strongly degenerate Lie p-algebra is the Witt algebra.

THEOREM B. *Let F be an algebraically closed field of characteristic $p > 5$. Let L be a simple Lie algebra of dimension p. Then L is either the p-dimensional Witt algebra or it is of classical type.*

Proof. Suppose first that L is not strongly degenerate. If L also satisfies condition (iii) of Theorem A, then all the hypotheses of this theorem are satisfied and L is of classical type. Hence, we suppose that condition (iii) is not satisfied. Pick a Cartan subalgebra H of L. The simplicity of L guarantees the existence of a nonzero root α. From the corresponding root space L_α choose $a \neq 0$. By assumption, there exists a root β and a b in L_β such that $bR_a^{p-1} \neq 0$. This implies that $L_\beta, L_{\beta-\alpha}, \ldots, L_{\beta+(p-1)\alpha}$ are distinct root spaces. Consequently, L has rank one and all the nonzero root spaces are also one-dimensional. Since $HR_a{}^2 \subseteq L_\alpha R_a = \{0\}$, we must have $\beta = 2\alpha$, and hence $[L_{-\alpha}L_\alpha] = \langle bR_a^{p-3}R_a\rangle = \langle bR_a^{p-2}\rangle \neq 0$. From the arbitrariness of α it follows that $[L_\gamma L_{-\gamma}] \neq 0$ for all nonzero roots γ. By appealing to the result of Block [1] stated in Section 5, we conclude that under these circumstances L is an Albert–Zassenhaus algebra, each of which is strongly degenerate. This contradiction shows that L is of classical type if it is not strongly degenerate.

Consider now the case when L is strongly degenerate. Recall the theorem stated in Section 2 guaranteeing the existence of a c in L such that $c \in L((p-3)/2)$. That is, $R_cR_x^{i-1}R_c = 0$ or $[cx^ic] = 0$ for all $i = 1, 2, \ldots, p-3$, and all $x \in L$. We claim that $[cx^{p-2}c] = \alpha c$ for all $x \in L$ and some $\alpha = \alpha(x) \in F$.

Assume $[ca^{p-2}c]$ and c are linearly independent for some a in L. It immediately follows that the set $\{c, [ca], \ldots, [ca^{p-2}], [ca^{p-2}c]\}$ forms a basis for L. If not, then they are linearly dependent. In particular, we have $\delta_0 c + \delta_1[ca] + \cdots + \delta_{p-1}[ca^{p-2}c] = 0$, where $\delta_i \in F$ and not all $\delta_i = 0$. Let j be the maximum subscript for which $\delta_i \neq 0$, $i < p-1$. By applying $R_a^{p-2-j}R_c$ to the left side of the equation and noting that $[ca^{p-2}ca^{p-2-j}c] = [ca^{p-2}]\, R_cR_a^{p-2-j}R_c = 0$ if $p-2-j \leqslant p-4$, we have $\delta_j[ca^{p-2}c] = 0$ if $j \geqslant 2$. This, of course, contradicts the linear independence of c and $[ca^{p-2}c]$. The same is also true if $j = 1$, since we have

$$\delta_1[ca^{p-2}c] + \delta_{p-1}[ca^{p-2}ca^{p-3}c] = 0$$

and

$$0 = [c[ca^{p-1}]\, a^{p-4}c] = \binom{p-1}{1} [ca^{p-2}ca^{p-3}c].$$

The second equation follows from the fact that for $x_1, \ldots, x_s$ in L [8, p. 198],

$$R_{x_1} \cdots R_{x_s} = \sum_{k_i < s} R_{z_i}^{k_i}$$

for some $z_i \in L$ if $s < p$. Since $\{c, [ca], \ldots, [ca^{p-2}], [ca^{p-2}c]\}$ forms a basis of L, we may write $a = \alpha c + \alpha_1[ca] + \cdots + \alpha_{p-2}[ca^{p-2}] + \alpha_{p-1}[ca^{p-2}c]$, or $-[ca] = \alpha_{p-2}[ca^{p-2}c]$. As this contradicts the linear independence of $[ca]$ and $[ca^{p-2}c]$, we conclude that $[cx^{p-2}c] = \alpha c$ for all x in L and some α in F.

It is easily seen that any element of the set $L(m)$ for $m > (p-3)/2$ would be a central element in L. Therefore, the set $L((p-1)/2)$ is empty, and there must exist an element a in L such that $[ca^{p-2}c] \neq 0$. For if $[cx^{p-2}c] = 0$ for all x in L, then, since $[cx^{p-1}c] = -[c[cx^{p-1}]] = -[cx^{p-1}c] - [cx^{p-2}cx]$, we would have $[cx^{p-2}c] = 0 \Leftrightarrow [cx^{p-1}c] = 0 \Leftrightarrow c \in L((p-1)/2)$. Pick such an a, normalized so that $[ca^{p-2}c] = -2c$. Then $2[ca^{p-1}c] = -[ca^{p-2}ca] = 2[ca]$, or $[ca^{p-1}c] = [ca]$. Obviously, the elements c, $[ca], \ldots, [ca^{p-2}]$ are linearly independent. If $[ca^{p-1}] = \alpha_0 c + \alpha_1[ca] + \cdots + \alpha_{p-2}[ca^{p-2}]$, then by applying R_c, we obtain the contradiction $[ca] = -2\alpha_{p-2}c$. Hence $L = \langle c, [ca], \ldots, [ca^{p-1}]\rangle$. The arguments to this point in the case where L is strongly degenerate are due to Kostrikin [8, p. 199].

Suppose H is a Cartan subalgebra of L. Consider the elements in L of the form $f_i = [c[ca^i]^{p-2}c]$. In order for f_i to be different from zero, we must have, when the $[ca^i]$'s are expanded, a term of the form $[ca^{l_1}ca^{l_2}c \cdots ca^{l_{p-1}}c]$, where $l_j \geq p-2$ for all $j = 1, 2, \ldots, p-1$. Thus, there must be a minimum of $(p-1)(p-2)$ a's present, implying $i = p-1$. Now there is at least one H-uniform element which, when expressed in terms of the $[ca^i]$'s and c, has a nonzero coefficient for $[ca^{p-1}]$. Otherwise, c, $[ca], \ldots, [ca^{p-2}]$ span L. If $d = \eta_0 c + \eta_1[ca] + \cdots + \eta_{p-1}[ca^{p-1}]$ is such an H-uniform element, then

$$\begin{aligned}[cd^{p-2}c] &= [c(\eta_0 c + \cdots + \eta_{p-1}[ca^{p-1}])^{p-2}c] \\ &= \eta_{p-1}^{p-2}[c[ca^{p-1}]^{p-2}c] \neq 0.\end{aligned}$$

Thus, we may assume that a is H-uniform, say $a \in L_\alpha$.

If $c \in H$, then it follows that $[ca] \in L_\alpha$, $[ca^2] \in L_{2\alpha}, \ldots, [ca^{p-1}] \in L_{(p-1)\alpha}$. However, $[ca^{p-2}c] = -2c$, which implies that $(p-2)\alpha = 0$, or $\alpha = 0$. We conclude $[ca^i] \in L_{i\alpha} = H$ for all $i = 0, 1, \ldots, p-1$, so $L = H$ is nilpotent, contradicting the simplicity of L.

Therefore, we may assume that $c = c_0 + c_1 + \cdots + c_n$, where $c_j \in L_{\gamma_j}$

for $j = 1, 2,..., n$, $c_0 \in H$ and $c_j \neq 0$ for some $j \neq 0$. We write $[ca^{p-1}] = [c_0a^{p-1}] + \cdots + [c_na^{p-1}]$. Since $[ca^{p-1}] \neq 0$, there is some j such that $[c_ja^{p-1}] \neq 0$, which implies that $\gamma_j, \gamma_j + \alpha,..., \gamma_j + (p-1)\alpha$ are roots. We first investigate the case when $\alpha \neq 0$. The above p roots are then distinct, and the root spaces are one-dimensional. In order that $[ca^{p-1}]$ be nonzero, it is necessary that $c_2 \in L_{\gamma_2} = L_{2\alpha}$ be nonzero. This being so, $[ca^{p-1}] = \omega a$ and hence $[ca^p] = 0$. From

$$[ca^{p-2}[ca^{p-1}]] = [ca^{p-2}ca^{p-1}] - \binom{p-1}{1}[ca^{p-1}ca^{p-2}]$$
$$= -2[ca^{p-1}] + [ca^{p-1}] = -[ca^{p-1}],$$

we conclude that $\omega = -1$, or $[ca^{p-1}] = -a$.

We now show that L is restricted; the result then follows from [8]. It is sufficient to show that $(\mathrm{ad}[ca^i])^p$ is inner for $i = 0, 1,..., p-1$. Expanding $[ca^j[ca^i]^p]$, we obtain terms of the form $d = [ca^{k_1}ca^{k_2} \cdots ca^{k_{p+1}}]$, where $0 \leqslant k_s \leqslant p-1$ ($[ca^p] = 0$). Since $c \in L((p-3)/2)$, d is zero if $\sum_{s=1}^{p} k_s < p(p-2)$, or, equivalently, $i < p-2$. For $i = p-2$, $[ca^j[ca^{p-2}]] = \sum_{r=0}^{p-2} (-1)^r \binom{p-2}{r}[ca^{j+r}ca^{p-2-r}] = \{-2\binom{p-2}{p-2-j} + \binom{p-2}{p-1-j}\}[ca^j] = \alpha[ca^j]$, $\alpha \in P$. Hence, $[ca^j[ca^{p-2}]^p] = \alpha^p[ca^j] = \alpha[ca^j]$. In the remaining case, $i = p-1$, $\sum_{s=1}^{p+1} k_s \geqslant p(p-1) = (p+1)(p-2) + 2$, implying that at least two k_s's are greater than $p-2$ and therefore $d = 0$ (since $[ca^p] = 0$). We therefore conclude that $(\mathrm{ad}[ca^i])^p = 0$ unless $i = p-2$, in which case $(\mathrm{ad}[ca^{p-2}])^p = \mathrm{ad}[ca^{p-2}]$.

It remains to examine the case $\alpha = 0$, that is, for a in H. Assuming this is so, set $a = \beta_0 c + \beta_1[ca] + \cdots + \beta_{p-1}[ca^{p-1}]$. Then

$$-2c = [ca^{p-2}c] = [c(\beta_0 c + \cdots + \beta_{p-1}[ca^{p-1}])^{p-2}c]$$
$$= \beta_{p-1}^{p-2}[c[ca^{p-1}]^{p-2}c],$$

or $\beta_{p-1} \neq 0$. If there is some other uniform element $f \notin H$ such that $f = \gamma_0 c + \cdots + \gamma_{p-1}[ca^{p-1}]$, where $\gamma_{p-1} \neq 0$, then, by the preceding argument, L is the Witt algebra. If not, every uniform element f contained in $\sum_{\gamma \neq 0} L_\gamma$ may be expressed in the form

$$f = \delta_0 c + \delta_1[ca] + \cdots + \delta_{p-2}[ca^{p-2}].$$

Thus, if $[ca^{p-1}] = d_0 + d_1$, where $d_0 \in H$ and $d_1 \in \sum_{\gamma \neq 0} L_\gamma$, then $d_0 \neq 0$. From this and the fact that $[ca^{p-1}] = [(c_0 + c_1 + \cdots + c_n)\, a^{p-1}] = [c_0a^{p-1}] + [(c_1 + \cdots + c_n)\, a^{p-1}]$, it follows that $[c_0a^{p-1}] = d_0 \neq 0$. Since c_0 and a are elements of H, we know that $[c_0a^t] = 0$ for some positive integer t, $[c_0a^{t-1}] \neq 0$. The elements $c_0, [c_0a],..., [c_0a^{t-1}]$ are linearly independent since otherwise $\delta_0 c_0 + \delta_1[c_0 a] + \cdots + \delta_{t-1}[c_0 a^{t-1}] = 0$ where not all the δ_i's are zero. If j is minimal with the property that $\delta_j \neq 0$, then applying R_a^{t-j-1} to the left side of the equation yields the contradiction $\delta_j[c_0 a^{t-1}] = 0$. Hence, c_0, $[c_0 a]$,..., $[c_0 a^{p-1}]$ are linearly independent over F and $[c_0 a^p] = 0$ since $\dim L = p$. This implies $L = H$, again contradicting the simplicity of L. Consequently, if L is strongly degenerate, it must be the Witt algebra, W_1. This concludes the proof of Theorem B.

7. Proof of Theorem C

The major portion of the proof of Theorem C involves proving the theorem under the added hypothesis that L is simple and that the characteristic of F is $p > 5$. Before beginning this we state the following result on Lie algebras of linear transformations, the proof of which appears in [11].

Lemma 7.1. *Let L be a Lie algebra of linear transformations on an n-dimensional vector space over a field F of characteristic $p > 0$. Then for any integer $r > 0$ and any U_1,..., U_r in L, the associative product $U_1 \cdots U_r$ can be written as a sum of powers of elements in L with exponents less than n plus a scalar multiple of the identity operator.*

Until stated otherwise, we assume L to be a simple Lie algebra over an algebraically closed field F of characteristic $p > 5$, which admits a faithful representation Γ of degree $n < p - 1$. Let V be the vector space upon which $\Gamma(L) = L^*$ acts, and let $\Gamma(x) = X$ for x in L.

Lemma 7.2. *If N is a nilpotent subalgebra of L and $\mathrm{ad}_L N$ is elementwise nilpotent, then the associative algebra generated by $\Gamma(N)$ is nilpotent.*

Proof. There exists at least one invariant irreducible subspace upon which Γ acts faithfully since the set of elements of $\Gamma(L)$ which annihilates a given invariant subspace forms an ideal and $\Gamma(L)$ is simple. Hence, by replacing V by an invariant subspace if necessary, we may assume Γ is irreducible.

Since $\Gamma(N) = N^*$ is nilpotent and the degree of $\Gamma < p$, we may apply Lie's theorem to N^* and obtain that the elements of N^* can be simultaneously put into upper (or lower) triangular form. For $\Gamma(s) = S \in L^*$ and $X \in N^*$ we have that $[SX^t] = 0$ for some positive integer t since $\mathrm{ad}_L\, x$ is nilpotent. Thus, the characteristic subspaces of X are invariant under S for any S in $\Gamma(L)$ [5, p. 40]. From the irreducibility of Γ we conclude that there exists only one characteristic subspace for X and hence for any element of N^*. The characteristic root for X must be zero as $X \in [L^*, L^*] = L^*$ (L^* is simple) implies $\operatorname{Tr} X = 0$. Thus, X is upper triangular with zeroes down the main diagonal for all X in N^*. In particular, if $X_i \in N^*$, $i = 1, 2,..., n$, then $X_1 X_2 \cdots X_n = 0$. This completes the proof.

LEMMA 7.3. *L is not strongly degenerate.*

Proof. Suppose L is strongly degenerate. By the theorem stated in Section 2 and by Proposition 2 of [9] there exists a nonzero c satisfying

$$R_c R_{x_1} \cdots R_{x_k} R_c = 0, \qquad 0 \leqslant k \leqslant p - 4,$$

and

$$R_c R_{x_1} \cdots R_{x_{p-3}} R_c R_{y_1} \cdots R_{y_{p-3}} R_c = 0,$$

with x_i, y_i in L. From the above and the formula

$$R_{[cu^k]} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} R_u{}^i R_c R_u^{k-i},$$

it follows that

$$R_{[cu_1^{k_1}]} R_{[cu_2^{k_2}]} \cdots R_{[cu_r^{k_r}]} = 0, \tag{6}$$

if $k_1 \leqslant p - 3,..., k_r \leqslant p - 3$ and $r > 2(p - 2)$.

In view of the simplicity of L, the ideal I_c generated by all products of the form $[cf_1 \cdots f_{i_1}]$ equals L, and $a = [cf_1 \cdots f_{i_1} c] \neq 0$, for some $f_1 ,..., f_{i_1}$ in L and $i_1 \geqslant p - 2$. Analogously, there are elements $g_1 ,..., g_{i_2}$ of L for which $b = [ag_1 \cdots g_{i_2} c] \neq 0$ since $I_a = L$. By repeating this process any preassigned number m times, we obtain

$$d = [cf_1 \cdots f_{i_1} c g_1 \cdots g_{i_2} c \cdots c h_1 \cdots h_{i_m} c] \neq 0.$$

Since $\Gamma(d) = [\Gamma(c)\, \Gamma(f_1) \cdots \Gamma(h_{i_m})\, \Gamma(c)] \neq 0$ is a linear combination of associative products of linear transformations, we must have

$$CU_1 \cdots U_{j_1} C \cdots CV_1 \cdots V_{j_{m-1}} CW_1 \cdots W_{j_m} C \neq 0$$

for some associative product where the $U,..., V, W$ are images under Γ of elements in the set $\{f_1 ,..., f_{i_1} ,..., h_1 ,..., h_{i_m}\}$. Appealing to Lemma 7.1, we write $W_1 \cdots W_{j_m}$ as a sum of powers Z^i, $Z \in L$, $i < n$. Replace $W_1 \cdots W_{j_m}$ in the above expression by an appropriate power $Z_m^{i_m}$ where $i_m < n$ is minimal so that the resulting expression $CU_1 \cdots CZ_m^{i_m} C$ is nonzero. From the minimality of i_m arises the equality

$$CU_1 \cdots CZ_m^{i_m} C = CU_1 \cdots [CZ_m^{i_m}] C.$$

In exactly the same manner we obtain Z_{m-1} so that $CV_1 \cdots V_{j_{m-1}}$ is replaced by $[CZ_{m-1}^{i_{m-1}}]$. After m such applications of Lemma 7.1, we have

$$(7) \qquad [CZ_1^{i_1}] \cdots [CZ_{m-1}^{i_{m-1}}][CZ_m^{i_m}] \neq 0,$$

where $i_j \leqslant n-1 \leqslant p-3$, $Z_j \in \Gamma(L)$, $j = 1,..., m$, and m is arbitrary. Obviously, not all the terms $[CZ_m^{i_j}]$ need be linearly independent since m is arbitrary. Let us now examine the subalgebra $L_0{}^* \subset \Gamma(L)$ generated by C, $[CZ_i^{i_1}],..., [CZ_m^{i_m}]$. All of its elements X can be written in the form $X = \Gamma(x)$, x belonging to the subalgebra L_0 of L generated by c, $[cz_i^{i_1}],..., [cz_m^{i_m}]$. From (6) and Engel's theorem it follows that L_0 is nilpotent, since, for x in L_0,

$$R_x^{2(p-1)} = \sum_{r \geqslant 2(p-1)} \alpha_{j_1 \ldots j_r} R_{[cz_{j_1}^{k_1}]} \cdots R_{[cz_{j_r}^{k_r}]} = 0.$$

From Lemma 7.2, it immediately follows that $L_0{}^*$ is associative nilpotent, contradicting (7) if $m \geqslant n$.

LEMMA 7.4. *L is of classical type.*

Proof. By Theorem A and the previous lemma, it is sufficient to show that for some Cartan subalgebra H of L there is an H-uniform element x satisfying the relation $R_x^{p-1} = 0$.

Assuming this to be false, we fix a Cartan subalgebra H and show that $R_a{}^k \neq 0$ for all H-uniform elements a where $k \geqslant 1$. If $R_a{}^k = 0$ for some $k(\geqslant p)$, then Lemma 7.2 can be applied to the one-dimensional subalgebra $N^* = \langle \Gamma(a) \rangle$, yielding that $A = \Gamma(a)$ is associative nilpotent, or $A^{p-2} = 0$. By assumption, there are H-uniform elements f, g in L such that $[fa^{p-1}] \neq 0$ and $[g[fa^{p-1}]^{p-1}] \neq 0$ (obviously, $[fa^{p-1}]$ is H-uniform). Furthermore, since Γ is a faithful representation,

$$[\Gamma(g)[\Gamma(f)\, \Gamma(a)^{p-1}]^{p-1}] \neq 0.$$

Expanding the left side to a linear combination of associative products, we must obtain a term of the form

$$A^{i_1}F \cdots FA^{i_k}GA^{i_{k+1}}F \cdots A^{i_p}FA^{i_{p+1}} \neq 0,$$

where $i_j < p-2$ since $A^{p-2} = 0$. However, $\sum_{j=1}^{p+1} i_j = (p-1)^2$ and $\sum_{j=1}^{p+1} i_j \leqslant (p+1)(p-3) < (p-1)^2$, proving that $R_a{}^k \neq 0$ for any positive integer k.

Fix $0 \neq u \in H$. The existence of a root γ such that $\gamma(u) \neq 0$ follows from the fact that $R_u{}^k \neq 0$ for any positive integer k and from the relation $zR_u^{p^s} = \alpha(u)^{p^s} z$ for z in L_α, α arbitrary, and some nonnegative integer

$s = s(z)$. Multiplying u by an appropriate constant, if necessary, we may assume $\gamma(u) = 1$. Thus, for any nonzero y in L_γ there is a nonnegative integer r such that if $\Gamma(y) = Y$, then $Y = Y(\text{ad } U)^{p^r} = Y \text{ad}(U^{p^r})$. Taking Y and U^{p^r} as linear transformations on V, we obtain a two-dimensional solvable Lie algebra to which Lie's theorem can be applied. In particular, it is obvious that if both Y and U^{p^r} are in triangular form and $[Y, U^{p^r}] = Y$, then Y is associative nilpotent. This contradicts the nonnilpotence of y, and the lemma is proved.

We now drop the assumptions that L is simple and $p > 5$ and assume that L is any Lie algebra satisfying the hypotheses of Theorem C. It is clear that $p > 3$ and $\dim L \leqslant (p-2)^2 - 1$. In fact, for $p = 5$ the result can be directly verified, and this case is eliminated from consideration.

To complete Theorem C for $p > 5$, we apply a recent result of Block on semisimple Lie algebras. Let $S_1, \ldots, S_r$ be simple algebras over F. Let $B_{n_i}(F)$ be the algebra of truncated polynomials in n_i indeterminants over F with $x_j{}^p = 0$ for each indeterminant x_j, $j = 1, 2, \ldots, n_i$. By der S_i and der $B_{n_i}(F)$ we denote the derivation algebras of S_i and $B_{n_i}(F)$, respectively. Theorem 9.3 of [2] states that if K is a semisimple Lie algebra over F, then it is a subalgebra of an algebra of the form

$$\oplus \sum_{i=1}^{r} \text{der } S_i \otimes B_{n_i}(F) + 1_{S_i} \otimes \text{der } B_{n_i}(F).$$

Moreover, we have that K contains $\oplus \sum_{i=1}^{r} \text{ad } S_i \otimes B_{n_i}(F)$ which is identified with $\oplus \sum_{i=1}^{r} S_i \otimes B_{n_i}(F)$ since S_i is centerless. By convention, $B_0(F) = F$.

Applying this result to L, it immediately follows that $n_i = 0$ or 1 for $i = 1, 2, \ldots, s$, since $\dim L < p^2$. Suppose that $n_i = 1$ for some i. Then

$$B_{n_i}(F) = \left\{ \sum_{j=0}^{p-1} a_j x^j \mid a_j \in F, x^p = 0 \right\}.$$

By the simplicity of S_i, we may pick $p - 1$ elements $s_1, s_2, \ldots, s_{p-1}$ from S_i so that $[s_1 \cdots s_{p-1}] \neq 0$. The Lie algebra N generated by $\{s_1 \otimes x, \ldots, s_{p-1} \otimes x\}$ is nilpotent since $x^p = 0$, and Lie's theorem is applicable to $\Gamma(N) = N^*$. If V is first decomposed into weight spaces relative to N^*, and then Lie's theorem is applied for each weight space individually, then the matrix corresponding to $\Gamma(s_i \otimes x)$ will have blocks down the main diagonal, each block being upper triangular of the form scalar $\cdot I +$ nilpotent. Since $n < p$, it follows that $0 = [(s_1 \otimes x) \cdots (s_{p-1} \otimes x)] = [s_1 \cdots s_{p-1}] \otimes x^{p-1}$, contradicting the choice of the s_i's. Hence, $n_i = 0$ for all $i = 1, 2, \ldots, r$, and since Γ acts faithfully on each S_i, by Lemma 7.4 they must all be of classical type. From the inequality $n < p$ and known properties of the simple classical algebras [11], it follows that each S_i possesses a nondegenerate trace form. This and a result of Zassenhaus [5, p. 74] imply that der $S_i = \text{ad } S_i$. This completes the proof of Theorem C.

Actually, the proof is valid for a more general class of Lie algebras. Let D be a set of derivations of L. An ideal I of L is said to be a D-ideal if it is invariant under D. If L contains no solvable D-ideals, then it is D-semisimple. Note that if $D = \text{ad}\, L$, then D-semisimple is equivalent to semisimple. The theorem of Block is proved for D-semisimple algebras. Consequently, the class of algebras considered in Theorem C can be enlarged to include these with no change in the conclusion.

We conclude with the statement of the following result, a direct consequence of Theorem C.

COROLLARY. *Let L be a semisimple Lie algebra, realized as a subset of the $n \times n$ matrices over F, $n < p - 1$. Then L is the direct sum of simple algebras of classical type.*

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Richard E. Block for his guidance.

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